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Configuration of the crucial set for a quadratic rational map

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Abstract

Let K be a complete, algebraically closed non-Archimedean valued field, and let $\varphi(z) \in K(z)$ have degree two. We describe the crucial set of φ in terms of the multipliers of φ at the classical fixed points, and use this to show that the crucial set determines a stratification of the moduli space $\mathcal{M}_2(K)$ related to the reduction type of φ. We apply this to settle a special case of a conjecture of Hsia regarding the density of repelling periodic points in the classical non-Archimedean Julia set.

Keywords: Crucial set, Quadratic map, Moduli space, Potential good reduction, Stratification

Mathematics Subject Classification: Primary 37P50, 11S82; Secondary 37P05, 11Y40

1 Background

Let K be an algebraically closed field, complete with respect to a non-Archimedean absolute value $|\cdot|_v$. Let $\mathcal{O} = \mathcal{O}_K$ denote its ring of integers and $\mathfrak{m} = \mathfrak{m}_K \subseteq \mathcal{O}$ its maximal ideal. Let $k = \mathcal{O}/\mathfrak{m}$ denote the residue field. We assume that $|\cdot|_v$ and the logarithm $\log_v$ are normalized so that $\mathrm{ord}_{\mathfrak{m}}(x) = -\log_v |x|_v$. We will typically drop the dependence on v and $\mathfrak{m}$ in the notation and simply write $|\cdot|$ and ord. Let $\mathbf{P}^1_K$ denote the Berkovich projective line over K; it is a compact, uniquely path connected Hausdorff space which contains $\mathbb{P}^1(K)$ as a dense subset.

Let $\varphi(z) \in K(z)$ be a rational map of degree $d \geq 2$. In [9], the third author showed there is a canonical way to assign non-negative integer weights $w_\varphi(P)$ to points in $\mathbf{P}^1_K$, for which the sum of the weights is $d-1$. The set of points which receive weight is called the *crucial set* of φ, and the probability measure $\nu_\varphi := \frac{1}{d-1}\sum_{P\in \mathbf{P}^1_K} w_\varphi(P)\delta_P$ is called the *crucial measure* of φ. When φ has potential good reduction, the crucial set consists of the single point where φ has good reduction. Otherwise, the crucial set appears to classify the type of bad reduction that φ has; this paper provides quantitative support for that idea when φ is a quadratic rational map.

The points in $\mathbf{P}^1_K$ are classified into four types, customarily labeled as type I through type IV. We will define these types in Sect. 2, but we mention here that the type I points are the "classical" points belonging to $\mathbb{P}^1(K)$, while the type II points are the non-classical points P at which φ has a meaningful reduction $\widetilde{\varphi}_P \in k(z)$. The crucial set is contained in the set of type II points ([9, Proposition 6.1]); in particular, it lies in $\mathbf{P}^1_K \backslash \mathbb{P}^1(K)$, and it is appropriate to talk about the reduction of φ at the points of the crucial set.

When φ is a quadratic rational map, the sum of the weights $w_\varphi(P)$ is $d-1=1$, and hence the crucial set consists of a unique point ξ. We can classify φ in terms of its reduction at ξ. Letting Γ_{Fix} denote the tree in $\mathbf{P}^1_K$ spanned by the type I fixed points of φ, we say that

- φ has *potential good reduction* if the reduction of φ at ξ is again a degree 2 map.
- φ has *potential additive reduction* if the reduction of φ at ξ is a degree 1 map conjugate to $z \mapsto z + \widetilde{a}$ for some $\widetilde{a} \in k \backslash \{\widetilde{0}\}$, and ξ is contained in Γ_{Fix}.
- φ has *potential multiplicative reduction* if the reduction of φ at ξ is a degree 1 map conjugate to $z \mapsto \widetilde{\lambda} z$ for some $\widetilde{\lambda} \in k \backslash \{\widetilde{0}, \widetilde{1}\}$, and ξ is a branch point of Γ_{Fix}.
- φ has *potential constant reduction* if the reduction of φ at ξ is a constant map, and ξ is a branch point of Γ_{Fix}.

In Sect. 2, we will explain why a quadratic rational map φ must satisfy exactly one of these conditions.

Our main result shows there is a relation between the reduction type of φ at ξ and the image of φ in $\mathcal{M}_2$, the moduli space of degree 2 rational maps. Using geometric invariant theory, Silverman [13] constructed $\mathcal{M}_d$ as a scheme over $\mathbb{Z}$ for all $d \geq 2$, and for $d=2$ showed there is a canonical isomorphism $\mathbf{s} : \mathcal{M}_2 \to \mathbb{A}^2$. (Milnor had shown this earlier over $\mathbb{C}$; see [8, Lemma 3.1].) This leads to a natural compactification of $\mathcal{M}_2$ as $\overline{\mathcal{M}_2} \cong \mathbb{P}^2$.

For a quadratic map $\varphi(z) \in K(z)$, let $[\varphi] \in \mathcal{M}_2(K)$ denote the point corresponding to the equivalence class of φ. We will base-change to $\mathcal{O}$ and regard $\mathcal{M}_2$ and $\overline{\mathcal{M}_2}$ as schemes over $\mathcal{O}$. The isomorphism $\mathbf{s} : \mathcal{M}_2 \to \mathbb{A}^2$ is given by $\mathbf{s}([\varphi]) = (\sigma_1(\varphi), \sigma_2(\varphi))$ where σ_1, σ_2 are the first and second symmetric functions in the multipliers at the fixed points of φ. We identify $\mathbb{A}^2(K)$ with $\{[x:y:1]\} \subset \mathbb{P}^2(K)$. Given a point $P \in \mathbb{P}^2(K)$, we write $\widetilde{P} \in \mathbb{P}^2(k)$ for the specialization of P modulo $\mathfrak{m}$.

For arbitrary $d \geq 2$, the connection between the crucial set and $\mathcal{M}_d$ was first noted in [9], where it was shown that points in the barycenter of ν_φ correspond to conjugates of φ having semi-stable reduction in the sense of geometric invariant theory. The main result of this paper is the following theorem, which says that for quadratic functions, the crucial set determines a stratification of $\mathcal{M}_2(K)$ compatible with specialization of $[\varphi]$ to $\overline{\mathcal{M}}_2(k)$:

Theorem 1.1 *Let K be a complete, algebraically closed non-Archimedean field, and let φ be a degree two rational map over K. Then*

(A) *φ has potential good reduction if and only if $\widetilde{\mathbf{s}([\varphi])} \in \mathbb{A}^2(k)$.*
(B) *φ has potential additive reduction if and only if $\widetilde{\mathbf{s}([\varphi])} = [\widetilde{1} : \widetilde{2} : \widetilde{0}]$.*
(C) *φ has potential multiplicative reduction if and only if $\widetilde{\mathbf{s}([\varphi])} = [\widetilde{1} : \widetilde{x} : \widetilde{0}]$ for some $\widetilde{x} \in k$ with $\widetilde{x} \neq \widetilde{2}$. In this case, $\widetilde{x} = \widetilde{\lambda} + \widetilde{\lambda}^{-1}$, where λ is the multiplier of an indifferent fixed point for φ.*
(D) *φ has potential constant reduction if and only if $\widetilde{\mathbf{s}([\varphi])} = [\widetilde{0} : \widetilde{1} : \widetilde{0}]$.*

Theorem 1.1 is proved by explicitly determining ξ and the reduction type of φ at ξ for a canonical representative of the equivalence class $[\varphi]$, and correlating these with the behavior of the multipliers of φ. If one is given an arbitrary quadratic rational map φ, it is possible to determine the unique point ξ in the crucial set of φ (as well as the reduction type of φ at ξ) by first conjugating to this canonical representative of $[\varphi]$ and then applying the results in Sect. 3. The fact that $\widetilde{\mathbf{s}([\varphi])} \in \mathbb{A}^2(k)$ if and only if φ has potential good reduction had previously been shown by D. Yap in her thesis [14, Theorem 3.0.3]; our theorem may be considered a refinement of Yap's result. One also notes the parallel between Theorem 1.1 and Milnor's description [8] of degenerations of quadratic maps over $\mathbb{C}$ as they approach the boundary of moduli space.

In Sect. 4.2, we also observe that part (A) of Theorem 1.1—and hence also the result of D. Yap cited above—implies the following result, which resolves a special case of a conjecture of L.-C. Hsia ([7, Conjecture 4.3]).

Proposition 1.2 *Let φ be a quadratic rational map defined over K, let $\mathcal{J}_\varphi(K) \subseteq \mathbb{P}^1(K)$ be the (classical) Julia set of φ, and let $\overline{\mathcal{R}_\varphi(K)}$ be the closure in $\mathbb{P}^1(K)$ of the set of type* I *repelling periodic points for φ. Then $\mathcal{J}_\varphi(K) = \overline{\mathcal{R}_\varphi(K)}$.*

Our method of proof for Proposition 1.2 is special to quadratic rational maps, and unfortunately cannot be extended to maps of arbitrary degree.

There is reason to believe that the connection between the configuration of the crucial set of a map φ and the location of $[\varphi]$ in moduli space should hold even in higher degrees. For example, Harvey, Milosevic, Rumely and Watson have given a classification theorem similar to Theorem 1.1 for cubic polynomials [6].

A significant difficulty which arises in higher degrees is that there are *many* more possible configurations for the crucial set: for a degree d map, there can be between 1 and $d-1$ points in the crucial set, and each point will exhibit one of four reduction types. One must also consider refinements to the reduction type that capture the geometric action of φ near a given point in the crucial set, and the relative locations of the different points in the crucial set. The number of configurations for a given d is clearly finite, but the problem of explicitly classifying the possible configurations for large d seems quite challenging. Nonetheless, the explicit connections in the case of quadratic rational maps and cubic polynomials suggest the possibility of a general theory linking the crucial set to the moduli space $\mathcal{M}_d$ for all d.

1.1 Outline of the paper

In Sect. 2, we introduce notation and concepts used in the rest of the article. In particular, we give a more detailed explanation of the weights w_φ for maps of arbitrary degree and the conditions under which a point can have weight. In Sect. 3 we relate the reduction type of the unique weighted point ξ to the multipliers at the classical fixed points. For this, we rely on two normal forms for quadratic rational maps given in [12]. In Sect. 4 we apply our analysis to prove Theorem 1.1, and give the application to Hsia's conjecture.

2 Notation and conventions

In this section we introduce terminology and notation used throughout the paper.

2.1 Berkovich space

Formally, the Berkovich affine line $\mathbf{A}^1_K$ over K is the collection of equivalence classes of multiplicative seminorms on $K[T]$ which extend the norm $|\cdot|$ on K. Berkovich [2, p. 18] showed that each such seminorm $[\cdot]_x$ corresponds to a decreasing sequence of closed discs $\{D(a_i, r_i)\}$ in K (more precisely, to a cofinal equivalence class of such sequences) via the correspondence

$$[f]_x := \lim_{i\to\infty} [f]_{D(a_i,r_i)}.$$

Here, $[f]_{D(a,r)} = \sup_{z\in D(a,r)} |f(z)|$ is the sup-norm on the disc $D(a,r)$. With this, we obtain a classification of the points of $\mathbf{A}^1_K$ into four types:

- *Points of type I* correspond to nested, decreasing sequences of discs $\{D(a_i, r_i)\}$ whose intersection is a single point in K; formally, these are the seminorms $[f]_a = |f(a)|$ for $a \in K$.
- *Points of type II* correspond to nested, decreasing sequences of discs whose intersection is a disc $D(a,r) \subseteq K$ with $r \in |K^\times|$. For polynomials $f \in K[T]$, we have $[f]_x = \sup_{z\in D(a,r)} |f(z)|$, and the supremum is achieved at some point in $D(a,r)$.
- *Points of type III* correspond to nested, decreasing sequences of discs whose intersection is a disc $D(a,r) \subseteq K$ with $r \notin |K^\times|$; as in the case of type II points, the corresponding seminorm is the sup-norm on $D(a,r)$. In this case, the supremum is not achieved in general.
- *Points of type IV* correspond to nested, decreasing sequences of discs $\{D(a_i, r_i)\}$ whose intersection is empty, but for which $\lim_{i\to\infty} r_i > 0$. Such points can occur only if the field K is not spherically complete.

One often describes points of type II and III in terms of their associated discs $D(a,r)$ using the shorthand $\zeta_{D(a,r)}$ or simply $\zeta_{a,r}$. The point corresponding to the unit disc is called the Gauss point (it corresponds to the Gauss norm on polynomials), and is written $\zeta_G = \zeta_{D(0,1)}$.

The construction of $\mathbf{P}^1_K$ from $\mathbf{A}^1_K$ is similar to the construction of $\mathbb{P}^1$ from $\mathbb{A}^1$, gluing two copies of $\mathbf{A}^1_K$ together by means of an involution of $\mathbf{A}^1_K\backslash\{0\}$; see Sect. 2.2 of [1] for details. We write $\mathbf{H}^1_K$ for $\mathbf{P}^1_K\backslash\mathbb{P}^1(K)$, the 'non-classical' part of $\mathbf{P}^1_K$.

The Berkovich projective line is typically endowed with the Berkovich-Gel'fand topology, which is the weakest topology for which the map $x \mapsto [f]_x$ is continuous for every $f \in K[T]$. In this topology, $\mathbf{P}^1_K$ is a compact Hausdorff space and is uniquely path connected. The points of type I are dense in $\mathbf{P}^1_K$ for this topology; so are the points of type II. In general the Berkovich-Gel'fand topology is not metrizable.

A rational map $\varphi \in K(z)$ induces a continuous action on $\mathbb{P}^1(K)$ by means of a lift $\Phi = [F, G]$, where $F, G \in K[X, Y]$ are homogeneous polynomials of degree $d = \deg(\varphi)$ such that $\varphi(z) = \frac{F(z,1)}{G(z,1)}$. This action extends continuously to all of $\mathbf{P}^1_K$, and preserves types of points. One can show that $\mathrm{PGL}_2(K)$ acts transitively on type II points, and that $\mathrm{PGL}_2(\mathcal{O})$ is the stabilizer of the Gauss point.

2.1.1 Tree structure

The Berkovich projective line has a canonical tree structure. The collection of points $\{\zeta_{a,r}\}_{r\in[t,s]} \subseteq \mathbf{P}^1_K$ is naturally homeomorphic to the real segment $[t, s]$. The type II points are dense along such a segment, and at any type II point there are infinitely many branches away from $\{\zeta_{a,r}\}_{r\in[t,s]}$; indeed the branches are in $1-1$ correspondence with the elements of $\mathbb{P}^1(k)$ for the residue field k. To illustrate this, consider the type II point $\zeta_G = \zeta_{D(0,1)}$. The branches off ζ_G come from equivalence classes of paths $[\zeta_G, x]$ sharing a common initial segment; these classes correspond to subdiscs $D(b,1)^- = \{x \in K : |x-b| < 1\}$

where $|b| \le 1$, and to the set $\mathbb{P}^1(K)\backslash D(0,1)$. Identifying $D(0,1)$ with the valuation ring $\mathcal{O}$, the subdiscs $D(b,1)^-$ are just the cosets $b+\mathfrak{m}$ in $k = \mathcal{O}/\mathfrak{m}$, and $\mathbb{P}^1(K)\backslash D(0,1)$ corresponds to $\widetilde{\infty} \in \mathbb{P}^1(k)$.

A more geometric way to think of the branches is in terms of tangent directions. Formally, a tangent direction $\mathbf{v}$ at P is an equivalence class of paths emanating from P. The collection of tangent directions at P will be denoted by T_P. For points of type II, T_P is in $1-1$ correspondence with $\mathbb{P}^1(k)$ as noted above. For points of type III, T_P consists of two directions, while for points of type I and IV, T_P consists of the unique direction pointing into $\mathbf{P}^1_K$. If P, Q are points of $\mathbf{P}^1_K$ with $\varphi(P) = Q$, there is a canonical induced surjective map $\varphi_* : T_P \to T_Q$.

2.1.2 Reduction of rational maps

The action of φ on the tangent space T_P is closely related to the notion of the reduction of φ, which we now describe. If $[F, G]$ is a lift of φ that has been scaled so that the coefficients all lie in $\mathcal{O}$, and so that at least one is a unit, we call $[F, G]$ a *normalized lift*, or *normalized representation*, of φ. Such a representation is unique up to scaling by a unit in $\mathcal{O}$. We can reduce each coefficient of a normalized lift $[F, G]$ modulo $\mathfrak{m}$. After removing common factors, we obtain a well-defined map $[\widetilde{F} : \widetilde{G}]$ on $\mathbb{P}^1(k)$. This map, called the reduction of φ at ζ_G, is denoted $\widetilde{\varphi}$.

If P is an arbitrary type II point, there is a $\gamma \in \mathrm{PGL}_2(K)$ for which $\gamma(\zeta_G) = P$; we define the reduction of φ at P to be the reduction of the conjugate $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$:

$$\widetilde{\varphi}_P(z) := \widetilde{\varphi^\gamma}(z).$$

The reduction $\widetilde{\varphi}_P$ is unique up to conjugation by an element of $\mathrm{PGL}_2(k)$; in particular, the degree $\deg(\widetilde{\varphi}_P)$ is well-defined.

It was shown by Rivera-Letelier that a type II point $P \in \mathbf{H}^1_K$ is fixed by φ if and only if the reduction $\widetilde{\varphi}_P$ is non-constant (see [1, Lemma 2.17]). Rivera-Letelier calls a type II point P a *repelling fixed point* if $\deg(\widetilde{\varphi}_P) \ge 2$, and he calls P an *indifferent fixed point* if $\deg(\widetilde{\varphi}_P) = 1$. The third author [9, Definition 2] gave a refined classification of indifferent fixed points in $\mathbf{H}^1_K$:

Definition 2.1 If P is a type II indifferent fixed point of φ, then after a change of coordinates on $\mathbb{P}^1(k)$, exactly one of the following holds:

- $\widetilde{\varphi}_P(z) = \widetilde{\lambda} z$ for some $\widetilde{\lambda} \in k\backslash\{\widetilde{0}, \widetilde{1}\}$, in which case we say P is a (Berkovich) multiplicatively indifferent fixed point for φ.
- $\widetilde{\varphi}_P(z) = z + \widetilde{a}$ for some $\widetilde{a} \in k\backslash\{\widetilde{0}\}$, in which case we say P is a (Berkovich) additively indifferent fixed point for φ.
- $\widetilde{\varphi}_P(z) = z$, in which case we say P is an id-indifferent fixed point for φ.

One should think of each of the above reduction types as describing the behavior of the map φ_* acting on T_P. More precisely, after conjugating φ by a suitable $\gamma \in \mathrm{PGL}_2(K)$ we can assume that $P = \zeta_G$ is fixed. Then $\widetilde{\varphi}$ is a well-defined non-constant map, and by making use of the identification $T_P \cong \mathbb{P}^1(k)$, if $\mathbf{v}_{\widetilde{a}} \in T_P$ corresponds to the point $\widetilde{a} \in \mathbb{P}^1(k)$, then $\varphi_*(\mathbf{v}_{\widetilde{a}}) = \mathbf{v}_{\widetilde{\varphi}(\widetilde{a})}$.

2.2 The crucial set

The crucial set was constructed in [9] and arose from the study of a certain function $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to \mathbb{R} \cup \{\infty\}$. The function $\mathrm{ordRes}_\varphi(\cdot)$ had been introduced in [10] to address the problem of finding conjugates φ^γ with minimal resultant. One obtains the crucial measure and crucial set by taking the graph-theoretic Laplacian of $\mathrm{ordRes}_\varphi(\cdot)$, restricted to a canonical tree $\Gamma_{\mathrm{FR}} \subset \mathbf{P}^1_K$.

In this section, we briefly recall this construction.

2.2.1 The function $\mathrm{ordRes}_\varphi(x)$

Let $\varphi \in K(z)$ have degree $d \geq 2$, and let $[F, G]$ be a lift of φ. Writing

$$F(X,Y) = a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d$$
$$G(X,Y) = b_0X^d + b_1X^{d-1}Y + \cdots + b_dY^d$$

put $\mathrm{ord}(F) = \min_{0\leq i\leq d}(\mathrm{ord}(a_i))$ and $\mathrm{ord}(G) = \min_{0\leq i\leq d}(\mathrm{ord}(b_i))$. If $\max(|a_i|, |b_i|) = 1$, (that is, $\min(\mathrm{ord}(F), \mathrm{ord}(G)) = 0$), the lift is normalized. For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(K)$, we define

$$F^\gamma(X,Y) := DF(AX+BY, CX+DY) - BG(AX+BY, CX+DY),$$
$$G^\gamma(X,Y) := -CF(AX+BY, CX+DY) + AG(AX+BY, CX+DY),$$

so that $\Phi^\gamma := [F^\gamma, G^\gamma]$ is a lift of φ^γ.

The resultant of the lift $\Phi = [F, G]$ is the determinant of the Sylvester matrix:

$$\mathrm{Res}(\Phi) := \mathrm{Res}(F,G) = \det \begin{pmatrix} a_0 & a_1 & \dots & a_{d-1} & a_d & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{d-1} & a_d & \dots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & a_0 & a_1 & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \dots & b_{d-1} & b_d & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{d-1} & b_d & \dots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & b_0 & b_1 & \dots & b_{d-1} & b_d \end{pmatrix}.$$

Let $\zeta \in \mathbf{P}^1_K$ be a type II point, and choose $\gamma \in \mathrm{PGL}_2(K)$ so that $\zeta = \gamma(\zeta_G)$. Fix a *normalized* lift Φ^γ of φ^γ. We then define

$$\mathrm{ordRes}_\varphi(\zeta) := \mathrm{ord}(\mathrm{Res}(\Phi^\gamma)).$$

Using standard formulas for the resultant from [11], one sees that $\mathrm{ordRes}_\varphi(\zeta)$ is well-defined, and that

$$\mathrm{ordRes}_\varphi(\zeta) = \mathrm{ordRes}_\varphi(\zeta_G) + (d^2+d)\,\mathrm{ord}\left(\det(\gamma)\right) - 2d\min\left(\mathrm{ord}(F^\gamma), \mathrm{ord}(G^\gamma)\right),$$

(The 'min' term assures we are using a normalized lift Φ^γ.) It is shown in [10] that the function ordRes_φ, which is *a priori* defined only on type II points, extends to a continuous function $\mathrm{ord\,Res}_\varphi : \mathbf{P}^1_K \to [0, \infty]$. For any $a \in \mathbb{P}^1(K)$, let $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ be the tree spanned by the classical fixed points of φ and the preimages of a under φ. It was shown in [10] that, regardless of the choice of $a \in \mathbb{P}^1(K)$, $\mathrm{ord\,Res}_\varphi$ attains its minimum value on $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$. In particular, $\mathrm{ordRes}_\varphi(\cdot)$ attains its minimum on the intersection of all such trees. The third author later showed in [9, Theorem 4.2] that this intersection is precisely the tree Γ_{FR} spanned by the classical fixed points and the repelling fixed points in $\mathbf{H}^1_K$; that is,

$$\Gamma_{\mathrm{FR}} = \bigcap_{a\in\mathbb{P}^1(K)} \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}. \tag{1}$$

This characterization is often useful for determining the tree Γ_{FR}.

2.2.2 The crucial measures

The crucial measure is obtained by taking the graph-theoretic Laplacian of $\mathrm{ordRes}_\varphi(\cdot)$ on (a suitable truncation[1] of) the tree Γ_{FR}. More precisely, if μ_{Br} is the 'branching measure' which gives each $P \in \Gamma_{\mathrm{FR}}$ the weight $1 - \frac{1}{2}v(P)$, where $v(P)$ is the valence of P in Γ_{FR}, then the crucial measure is defined as follows:

Definition 2.2 (Rumely [9, Corollary 6.5]) The crucial measure associated to φ is the measure ν_φ on Γ_{FR} defined by

$$\Delta_{\Gamma_{\mathrm{FR}}}(\mathrm{ordRes}_\varphi(\cdot)) = 2(d^2 - d)(\mu_{Br} - \nu_\varphi).$$

It is a probability measure with finite support, and its support is contained in the set of type II points.

The crucial measure is canonically attached to φ, because the function $\mathrm{ordRes}_\varphi(\cdot)$ and the tree Γ_{FR} are canonical. It is a conjugation equivariant of φ in $\mathbf{H}^1_K$, just as the sets of classical fixed points and critical points are conjugation equivariants in $\mathbb{P}^1(K)$.

The third author gave an explicit expression for ν_φ as a sum of weighted point masses:

$$\nu_\varphi = \frac{1}{d-1} \sum_{P\in \mathbf{P}^1_K} w_\varphi(P)\,\delta_P,$$

where the weights $w_\varphi(P)$ are as follows [9, Definition 8]:

Definition 2.3 For a point $P \in \mathbf{P}^1_K$, if P is fixed by φ, let $N_{\mathrm{shearing},\varphi}(P)$ be the number of directions $\mathbf{v} \in T_P$ that contain type I fixed points but are moved by φ_*. Let $v(P)$ denote the valence of P in Γ_{FR} (set $v(P) = 0$ if $P \notin \Gamma_{\mathrm{FR}}$). Then the weight $w_\varphi(P)$ of a point $P \in \mathbf{P}^1_K$ is as follows:

(A) If P is a type II fixed point of φ, then $w_\varphi(P) = \deg(\widetilde{\varphi}_P) - 1 + N_{\mathrm{shearing},\varphi}(P)$.
(B) If P is a branch point of Γ_{Fix} which is moved by φ (necessarily of type II), then $w_\varphi(P) = v(P) - 2$.
(C) Otherwise, $w_\varphi(P) = 0$.

The fact that ν_φ is a probability measure is equivalent to the following formula:

Theorem 2.4 (Rumely [9, Theorem 6.2]) *Let $\varphi \in K(z)$ have degree $d \geq 2$. Then*

$$\sum_{P\in\mathbf{P}^1_K} w_\varphi(P) = d - 1. \tag{2}$$

[1] In order to apply the theory of graph Laplacians, one must first 'prune' the tree Γ_{FR} to remove its type I endpoints—see [9, p.25]. We omit the details here, as they won't be necessary in this article.

We emphasize that for a quadratic rational map, formula (2) implies there is a unique point $\xi \in \mathbf{P}^1_K$ with $w_\varphi(\xi) > 0$. Moreover, it follows from [9, Proposition 6.1], that when φ is quadratic, a point ξ has positive weight in precisely the following four situations:

- $\widetilde{\varphi}_\xi$ *has degree 2*: Here, ξ is a repelling fixed point for φ, and $w_\varphi(\xi) > 0$ by part (A) of Definition 2.3. In this case, φ has potential good reduction.
- $\widetilde{\varphi}_\xi$ *is conjugate to* $z \mapsto z + \widetilde{a}$ *for some* $\widetilde{a} \in k\backslash\{\widetilde{0}\}$, and ξ *is contained in* Γ_{Fix}: Here, ξ is an additively indifferent fixed point with at least two tangent directions containing type I fixed points. Since φ_* fixes only one tangent direction, ξ must have a shearing direction, hence $w_\varphi(\xi) > 0$ by Definition 2.3 (A). In this case, φ has potential additive reduction. (Note that since $w_\varphi(\xi) = 1$, ξ can have only one shearing direction; thus ξ cannot be a branch point of Γ_{Fix}.)
- $\widetilde{\varphi}_\xi$ *is conjugate to* $z \mapsto \widetilde{\lambda} z$ *for some* $\widetilde{\lambda} \in k\backslash\{\widetilde{0}, \widetilde{1}\}$, *and* ξ *is a branch point of* Γ_{Fix}: Here, ξ is a multiplicatively indifferent fixed point, but now ξ has three tangent directions containing type I fixed points. Since φ_* fixes only two tangent directions, ξ must have a shearing direction, so $w_\varphi(\xi) > 0$—once again, by part (A) of Definition 2.3. In this case, φ has potential multiplicative reduction
- $\widetilde{\varphi}_\xi$ *is constant, and* ξ *is a branch point of* Γ_{Fix}: That $w_\varphi(\xi) > 0$ follows immediately from part (B) of Definition 2.3. In this case, φ has potential constant reduction.

To illustrate the complications that arise in classifying reduction types for higher degree maps, we give here two examples of crucial sets for cubic polynomials, taken from the forthcoming work [6]:

- Let $\varphi(z) = z(z - F_1)(z - F_2) + z$, where $1 < |F_2|$ and $1/|F_2| < |F_1| < |F_2|$. Then the points $\zeta_{D(0,|F_1|)}$ and $\zeta_{D(0,|F_2|)}$ are branch points of the tree Γ_{Fix} with valence 3 which are moved by φ. Thus, $w_\varphi(\zeta_{D(0,|F_1|)}) = w_\varphi(\zeta_{D(0,|F_2|)}) = 3 - 2 = 1$. By the weight formula (2), these are the only points in the crucial set.
- Let $\varphi(z) = z(z - F_1)(z - F_2) + z$ be as above, where $1 < |F_2|$ but now $|F_1| \leq 1/|F_2|$. Then $\zeta_{D(0,|F_2|)}$ and $\zeta_{D(0,1/|F_2|)}$ are the points in the crucial set, each of weight 1: The point $\zeta_{D(0,|F_2|)}$ is a branch point of Γ_{Fix} with valence 3 that is moved by φ, hence its weight is $w_\varphi(\zeta_{D(0,|F_2|)}) = 3 - 2 = 1$. The point $\zeta_{D(0,1/|F_2|)}$, however, is fixed by φ, but has no shearing—here, the reduction of φ at $\zeta_{D(0,1/|F_2|)}$ is a quadratic polynomial. Thus $w_\varphi\left(\zeta_{D(0,1/|F_2|)}\right) = 1$ as well.

There are, in all, five configurations of the crucial set that can occur for cubic polynomials; the interested reader is directed to [6] for details. For general cubic rational functions, one can identify at least 19 potential configurations, though it is not yet known if each of these actually occurs.

2.3 The moduli space of quadratic rational maps

Using geometric invariant theory, Silverman [13, Theorem 1.1] constructed the moduli space $\mathcal{M}_d$ for rational maps of degree $d \geq 2$ as a scheme over $\mathbb{Z}$.

If $\alpha \in K$ is a fixed point for the rational map φ, the derivative $\varphi'(\alpha)$ is called the multiplier of φ at α. It is well-known that the multiplier is independent of the choice of coordinates, which means the multiplier at φ at $\infty \in \mathbb{P}^1(K)$ can be defined by changing coordinates.

Silverman showed [13, Theorem 5.1] there is a natural isomorphism $\mathbf{s} : \mathcal{M}_2 \to \mathbb{A}^2$ as schemes over $\mathbb{Z}$. More precisely, he showed that the first and second elementary symmetric functions σ_1, σ_2 of the multipliers at the fixed points give coordinates on $\mathcal{M}_2$.

This means that if $\varphi(z) \in K(z)$ is a quadratic map with fixed points $\alpha_1, \alpha_2, \alpha_3$ (listed with multiplicity) and corresponding multipliers $\lambda_1, \lambda_2, \lambda_3$, and if we put

$$\sigma_1(\varphi) = \lambda_1 + \lambda_2 + \lambda_3, \qquad \sigma_2(\varphi) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$

then the point $[\varphi]$ in $\mathcal{M}_2(K) \cong \mathbb{A}^2(K)$ corresponding to φ is $(\sigma_1(\varphi), \sigma_2(\varphi))$.

3 The crucial sets of quadratic maps

The behavior of a rational map φ near a classical fixed point $\alpha \in \mathbb{P}^1(K)$ is governed by the multiplier at α. In this section, we explicitly describe the crucial set for quadratic rational maps in terms of the multipliers at the classical fixed points.

Letting λ be the multiplier at α, one says that α is

attracting, if $|\lambda| < 1$;

indifferent, if $|\lambda| = 1$; and

repelling, if $|\lambda| > 1$.

Throughout this section, we will let $\alpha_1, \alpha_2, \alpha_3$ be the (not necessarily distinct) fixed points for φ, and we will let $\lambda_1, \lambda_2, \lambda_3$ be the corresponding multipliers.

3.1 Maps with a multiple fixed point

We begin our classification by considering quadratic rational maps φ with a multiple fixed point. In this case, we may assume without loss of generality that $\alpha_1 = \alpha_2$, which means that necessarily $\lambda_1 = \lambda_2 = 1$. By [12, Lemma 2.46], φ is conjugate to the rational map

$$z \mapsto z + \sqrt{1-\lambda_3} + \frac{1}{z},$$

with fixed points $\alpha_1 = \alpha_2 = \infty$ and $\alpha_3 = -\frac{1}{\sqrt{1-\lambda_3}}$, and multipliers $\lambda_1 = \lambda_2 = 1$ and λ_3.

Proposition 3.1 *Suppose*

$$\varphi(z) = z + \sqrt{1-\lambda_3} + \frac{1}{z},$$

and let ξ be the unique point in $\mathbf{P}^1_K$ with $w_\varphi(\xi) = 1$.

(A) *If $|\lambda_3| \le 1$, then $\xi = \zeta_G$ and φ has (potential) good reduction.*
(B) *If $|\lambda_3| > 1$, then $\xi = \zeta_{D(0,\sqrt{|\lambda_3|})}$ and φ has potential additive reduction.*

Proof If $|\lambda_3| \le 1$, then $|1-\lambda_3| \le 1$ so $\varphi(z)$ has good reduction at ζ_G, proving (A). We therefore suppose that $|\lambda_3| > 1$. Using (1) we find that

$$\Gamma_{\mathrm{FR}} = \Gamma_{\mathrm{Fix}} = \left[-1/\sqrt{1-\lambda_3}, \infty\right].$$

In particular, since $|\lambda_3| > 1$, we have $\zeta_{D(0,\sqrt{|\lambda_3|})} \in \Gamma_{\mathrm{Fix}}$. It remains to show that $\zeta_{D(0,\sqrt{|\lambda_3|})}$ is an additively indifferent fixed point for φ.

Let $\gamma(z) = \sqrt{1-\lambda_3} \cdot z$; since $|\lambda_3| > 1$, we have $|\sqrt{1-\lambda_3}| = \sqrt{|\lambda_3|}$, so $\gamma(\zeta_G) = \zeta_{D(0,\sqrt{|\lambda_3|})}$. Conjugating φ by γ we find that

$$\varphi^\gamma(z) = \frac{z^2 + z + 1/(1-\lambda_3)}{z}.$$

Since $|1-\lambda_3| = |\lambda_3| > 1$, reducing modulo $\mathfrak{m}$ yields

$$\widetilde{\varphi}_{\zeta_{D(0,\sqrt{|\lambda_3|})}} = \widetilde{\varphi^\gamma}(z) = \frac{z^2+z}{z} = z + \widetilde{1}.$$

Therefore $\zeta_{D(0,\sqrt{|\lambda_3|})}$ is the unique weighted point, and φ has potential additive reduction. □

3.2 Maps with distinct fixed points

We now turn to quadratic rational maps with three distinct classical fixed points. In this case, the multiplier of the third fixed point is determined by the multipliers of the other two:

Lemma 3.2 *Let φ be a degree two rational map with three distinct classical fixed points. Let λ_1 and λ_2 be the multipliers of two of the fixed points. Then the third has multiplier*

$$\lambda_3 = \frac{\lambda_1+\lambda_2-2}{\lambda_1\lambda_2-1}. \tag{3}$$

Proof Since φ has three distinct fixed points, none of the multipliers can be equal to one. Therefore, we have the well-known formula (see [11, Theorem 1.14])

$$\frac{1}{1-\lambda_1} + \frac{1}{1-\lambda_2} + \frac{1}{1-\lambda_3} = 1. \tag{4}$$

Solving for λ_3 yields the desired result. □

Lemma 3.3 *Let φ be a degree two rational map over K with three distinct classical fixed points. Then these cannot all be repelling. Moreover,*

(A) *if φ has two classical repelling fixed points, then the third is attracting*;
(B) *if φ has only one classical repelling fixed point, then the other two are indifferent*;
(C) *if φ has no classical repelling fixed points, then either some pair of multipliers satisfies $\widetilde{\lambda_i\lambda_j} \neq \widetilde{1}$, or else $\widetilde{\lambda_1} = \widetilde{\lambda_2} = \widetilde{\lambda_3} = \widetilde{1}$.*

Proof Suppose that φ has two repelling fixed points, say α_1 and α_2, with multipliers λ_1 and λ_2. By (3), the multiplier λ_3 of α_3 satisfies

$$|\lambda_3| = \frac{|\lambda_1+\lambda_2-2|}{|\lambda_1\lambda_2-1|} \;\leq\; \frac{\max\{|\lambda_1|, |\lambda_2|\}}{|\lambda_1\lambda_2|} \;<\; 1,$$

so α_3 is attracting. This shows that φ cannot have three repelling fixed points, and also proves (A).

To show (B), suppose the fixed points of φ are labeled so that

$$|\lambda_1| \;>\; 1 \;\geq\; |\lambda_2| \;\geq\; |\lambda_3|.$$

Again using (3), we have

$$|\lambda_3| = \frac{|\lambda_1+\lambda_2-2|}{|\lambda_1\lambda_2-1|} \;\geq\; \frac{|\lambda_1|}{\max\{|\lambda_1\lambda_2|, 1\}} \;\geq\; \frac{|\lambda_1|}{\max\{|\lambda_1|, 1\}} = 1.$$

Since we assumed that $|\lambda_3| \leq 1$, equality holds throughout, and therefore $|\lambda_2| = |\lambda_3| = 1$.

To show (C), suppose that $|\lambda_1|, |\lambda_2|, |\lambda_3| \leq 1$. If some pair of multipliers satisfies $\widetilde{\lambda_i\lambda_j} \neq \widetilde{1}$, we are done. Otherwise $\widetilde{\lambda_1\lambda_2} = \widetilde{\lambda_1\lambda_3} = \widetilde{\lambda_2\lambda_3} = \widetilde{1}$. Considering these equalities in pairs, we conclude there is a $\widetilde{c} \in k$ such that $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{c}$. In particular, we have $\widetilde{c}^2 = \widetilde{1}$, so $\widetilde{c} \in \{\pm\widetilde{1}\}$. If $\widetilde{c} = \widetilde{1}$ (which is necessarily true if $\mathrm{char}(k) = 2$), then we are done, so assume that $\mathrm{char}(k) \neq 2$ and $\widetilde{c} = -\widetilde{1}$. In this case, reducing (4) modulo $\mathfrak{m}$ yields the equation $\widetilde{(3/2)} = \widetilde{1}$, which implies that $\widetilde{2} = \widetilde{3}$, a contradiction. Hence $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{1}$. □

Our next result is parallel to Proposition 3.1 and describes the structure of the crucial set for a quadratic rational map with three distinct fixed points. For such maps, none of the multipliers can be 1, and we know from [12, Lemma 2.46] that φ is conjugate to a map of the form

$$z \mapsto \frac{z^2 + \lambda_1 z}{\lambda_2 z + 1},$$

where λ_1 and λ_2 are two of the fixed point multipliers for φ. We will henceforth assume φ is given in this form. The fixed points of φ are then $\alpha_1 = 0$, $\alpha_2 = \infty$, and $\alpha_3 = (\lambda_1 - 1)/(\lambda_2 - 1)$, with multipliers λ_1, λ_2, and λ_3, respectively. This means that Γ_{Fix} has a single branch point at $\zeta_{D(0,|\alpha_3|)}$.

Furthermore, if φ has no repelling classical fixed points, i.e., if $|\lambda_1|, |\lambda_2|, |\lambda_3| \le 1$, then by Lemma 3.3 either $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{1}$ or, by permuting the fixed points via an additional conjugation if necessary, $\widetilde{\lambda_1\lambda_2} \neq \widetilde{1}$. On the other hand, if φ has a repelling classical fixed point, then by Lemma 3.3 it also has a non-repelling classical fixed point; permuting the fixed points via a conjugation of φ if necessary, we can assume that $|\lambda_1| > 1 \ge |\lambda_2|$.

Proposition 3.4 *Let*

$$\varphi(z) = \frac{z^2 + \lambda_1 z}{\lambda_2 z + 1},$$

and let ξ be the unique point in $\mathbf{P}^1_K$ with $w_\varphi(\xi) = 1$.

(A) *If φ has no repelling classical fixed points, then φ has (potential) good reduction. Permuting the indices via an additional conjugation of φ if necessary, we can assume that either $\widetilde{\lambda_1\lambda_2} \neq 1$, or that $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{1}$. With this ordering of the indices, we have*

   (i) *if $\widetilde{\lambda_1\lambda_2} \neq \widetilde{1}$, then $\xi = \zeta_G$;*
   (ii) *if $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{1}$, then $\xi = \zeta_{D(-1,\sqrt{|\lambda_1\lambda_2 - 1|})}$.*

(B) *Suppose φ has at least one repelling classical fixed point, hence also a non-repelling fixed point by* Lemma 3.3. Permuting the indices via an additional conjugation of φ if necessary, we can assume that $|\lambda_1| > 1 \ge |\lambda_2|$. With this ordering of the indices, we find that $\xi = \zeta_{D(0,|\lambda_1|)}$. Moreover,

   (i) if $\widetilde{\lambda}_2 \notin \{\widetilde{0}, \widetilde{1}\}$, then φ has potential multiplicative reduction.
   (ii) if $\widetilde{\lambda}_2 = \widetilde{1}$, then φ has potential additive reduction.
   (iii) if $\widetilde{\lambda}_2 = \widetilde{0}$, then φ has potential constant reduction.

Remark Note that in case (A)(ii), the point ξ is different from ζ_G, since $|\lambda_1\lambda_2 - 1| < 1$ and therefore

$$D\left(-1, \sqrt{|\lambda_1\lambda_2 - 1|}\right) \subsetneq D(-1, 1) = D(0, 1).$$

For a quadratic φ, this is the only situation where Γ_{FR} can be strictly larger than Γ_{Fix} .

Proof First, assume that φ has no classical repelling fixed points, which means that each of the multipliers lies in $\mathcal{O}$. In particular, this implies that the expression for φ given in the proposition is already normalized. Since $\mathrm{Res}(\Phi) = 1 - \lambda_1\lambda_2$, we see that if $\widetilde{\lambda_1\lambda_2} \neq \widetilde{1}$, then $|\mathrm{Res}(\Phi)| = |1 - \lambda_1\lambda_2| = 1$, so φ has good reduction at ζ_G, proving (A)(i).

Now suppose that $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \widetilde{\lambda}_3 = \widetilde{1}$. This means that $|\lambda_1\lambda_2 - 1| < 1$. Set $\rho := \sqrt{\lambda_1\lambda_2 - 1}$, and let $r := |\rho| < 1$. Put $\gamma(z) := \rho z - 1$, so that $\gamma(\zeta_G) = \zeta_{D(-1,r)}$. To prove (A)(ii), it suffices to show that φ^γ has good reduction.

The map φ^γ is given by

$$\varphi^\gamma(z) = \frac{\rho^2 z^2 + \rho(\lambda_1 + \lambda_2 - 2)z - (\lambda_1 + \lambda_2 - 2)}{\rho^2\lambda_2 z - \rho(\lambda_2 - 1)}.$$

Since $\lambda_1, \lambda_2 \in \mathcal{O}$ by assumption, all of the coefficients of φ^γ belong to $\mathcal{O}$. However, as we will now see, in fact all of the coefficients of φ^γ lie in $\mathfrak{m}$, so we need a normalized representation of φ^γ.

Since $|\lambda_3| = 1$, it follows from (3) that

$$|\lambda_1 + \lambda_2 - 2| = |\lambda_1\lambda_2 - 1| = |\rho^2| = r^2 < 1.$$

We also claim that $|\lambda_2 - 1| \le r$. Indeed, suppose to the contrary that $|\lambda_2 - 1| > r$. Then

$$|(\lambda_2 - 1)^2| > |\lambda_1\lambda_2 - 1| = |\lambda_1 + \lambda_2 - 2|,$$

so that

$$\begin{aligned} |(\lambda_2 - 1)^2| > |\lambda_1 + \lambda_2 - 2| &= |\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_2| \\ &= |(\lambda_2 - 1)^2 + (\lambda_1\lambda_2 - 1)| = |(\lambda_2 - 1)^2|, \end{aligned}$$

a contradiction.

We now see that the absolute values of the coefficients of φ^γ are as follows:

$$\begin{aligned} |\rho^2| = |-(\lambda_1 + \lambda_2 - 2)| = |\rho^2\lambda_2| &= r^2; \\ |\rho(\lambda_1 + \lambda_2 - 2)| &= r^3; \\ |\rho(\lambda_2 - 1)| &\le r^2; \end{aligned}$$

so the maximum among the absolute values of the coefficients is r^2. We therefore divide all coefficients by ρ^2 to obtain the normalized representation

$$\varphi^\gamma(z) = \frac{z^2 + \frac{\lambda_1+\lambda_2-2}{\rho} z - \frac{\lambda_1+\lambda_2-2}{\rho^2}}{\lambda_2 z - \frac{\lambda_2-1}{\rho}}.$$

The resultant of the natural lift Φ^γ is

$$\mathrm{Res}(\Phi^\gamma) = -\frac{\lambda_1\lambda_2 - 1}{\rho^2} = -1,$$

and therefore φ^γ has good reduction. It follows that $\gamma(\zeta_G) = \zeta_{D(-1,r)}$ is a repelling fixed point for φ, and hence $\xi = \zeta_{D(-1,r)}$, as claimed.

To prove (B), take $\gamma(z) := \lambda_1 z$, so that $\gamma(\zeta_G) = \zeta_{D(0,|\lambda_1|)}$. Then

$$\varphi^\gamma(z) = \frac{z^2 + z}{\lambda_2 z + 1/\lambda_1}.$$

Since $|\lambda_1| > 1 \ge |\lambda_2|$, the obvious lift Φ^γ is normalized, and we can reduce modulo $\mathfrak{m}$:

$$\widetilde{\varphi}_{\zeta_{D(0,|\lambda_1|)}} = \widetilde{\varphi^\gamma}(z) = \frac{z^2 + z}{\widetilde{\lambda}_2 z} = \frac{\widetilde{1}}{\widetilde{\lambda}_2}(z + \widetilde{1}).$$

We now consider several cases:

- If $\widetilde{\lambda_2} \notin \{\widetilde{0}, \widetilde{1}\}$, then $\widetilde{\varphi^\gamma}$ is conjugate to the map $(\widetilde{1}/\widetilde{\lambda_2})z$ via $z \mapsto z + \widetilde{1}/(\widetilde{\lambda_2} - \widetilde{1})$; hence $\zeta_{D(0,|\lambda_1|)}$ is a multiplicatively indifferent fixed point for φ. Moreover,

$$|\alpha_3| = \frac{|\lambda_1 - 1|}{|\lambda_2 - 1|} = |\lambda_1|,$$

so $\zeta_{D(0,|\lambda_1|)} = \zeta_{D(0,|\alpha_3|)}$ is a branch point of Γ_{Fix}. We conclude that φ has potential multiplicative reduction, thereby proving (B)(i).
- If $\widetilde{\lambda_2} = \widetilde{1}$, then $\widetilde{\varphi^\gamma}(z) = z + \widetilde{1}$, thus $\zeta_{D(0,|\lambda_1|)}$ is an additively indifferent fixed point for φ. Moreover, since

$$|\alpha_3| = \frac{|\lambda_1 - 1|}{|\lambda_2 - 1|} > |\lambda_1|,$$

$\zeta_{D(0,|\lambda_1|)}$ lies on the segment $[0, \zeta_{D(0,|\alpha_3|)}] \subset [0, \alpha_3] \subset \Gamma_{\mathrm{Fix}}$. Therefore φ has potential additive reduction, proving (B)(ii).
- Finally, if $\widetilde{\lambda_2} = \widetilde{0}$, then $\widetilde{\varphi^\gamma}$ is the constant map $\widetilde{\infty}$. This means that $\zeta_{D(0,|\lambda_1|)}$ is not a fixed point under φ. Arguing just as we did for (B)(i), we see that $\zeta_{D(0,|\lambda_1|)}$ is the branch point for Γ_{Fix}. Therefore φ has potential constant reduction, which proves (B)(iii).

□

4 Applications to moduli space

4.1 Proof of the main theorem

We are now ready to prove our main result, Theorem 1.1, which says that for quadratic maps the crucial set gives a stratification of $\mathcal{M}_2(K)$ compatible with specialization to $\overline{\mathcal{M}}_2(k)$. Recall that $\mathcal{M}_2 \cong \mathbb{A}^2$ and $\overline{\mathcal{M}}_2 \cong \mathbb{P}^2$ as schemes over $\mathbb{Z}$; by abuse of notation we view the isomorphism $\mathbf{s} = (\sigma_1, \sigma_2) : \mathcal{M}_2(K) \to \mathbb{A}^2(K)$ as an embedding

$$\mathbf{s} = [\sigma_1 : \sigma_2 : 1] : \mathcal{M}_2(K) \hookrightarrow \mathbb{P}^2(K).$$

Finally, for a point $P \in \mathbb{P}^2(K)$, we denote by $\widetilde{P} \in \mathbb{P}^2(k)$ the specialization of P modulo $\mathfrak{m}$.

Proof of Theorem 1.1 Since φ has exactly one of the reduction types given in the statement of the theorem, and since $\mathbb{P}^2(k)$ is equal to the disjoint union

$$\mathbb{P}^2(k) = \mathbb{A}^2(k) \sqcup \left\{[\widetilde{1} : \widetilde{x} : \widetilde{0}] \mid \widetilde{x} \in k, \widetilde{x} \neq \widetilde{2}\right\} \sqcup \{[\widetilde{1} : \widetilde{2} : \widetilde{0}]\} \sqcup \{[\widetilde{0} : \widetilde{1} : \widetilde{0}]\},$$

it suffices to prove only the forward implications of the statements in the theorem.

First, suppose φ has potential good reduction. If φ has three distinct fixed points, it follows from Proposition 3.4 that φ has no repelling fixed points. Thus all of the multipliers of φ lie in $\mathcal{O}$. In particular, this means that $\sigma_1, \sigma_2 \in \mathcal{O}$, so

$$\widetilde{\mathbf{s}([\varphi])} = [\widetilde{\sigma_1} : \widetilde{\sigma_2} : \widetilde{1}] \in \mathbb{A}^2(k).$$

If, on the other hand, φ has a multiple fixed point, then for a suitable ordering of the multipliers we have $\lambda_1 = \lambda_2 = 1$, and by Proposition 3.1 we find $|\lambda_3| \le 1$ as well. Once again, all the multipliers of φ lie in $\mathcal{O}$, and so

$$\widetilde{\mathbf{s}([\varphi])} = [\widetilde{\sigma_1} : \widetilde{\sigma_2} : \widetilde{1}] \in \mathbb{A}^2(k).$$

If φ does not have potential good reduction, we claim that it must have at least one repelling classical fixed point and one non-repelling classical fixed point. Indeed, in the case that φ has a multiple fixed point, this follows from Proposition 3.1; in the case that φ has three distinct fixed points, we know from Proposition 3.4 that φ must have at least one classical repelling fixed point, in which case φ also has a non-repelling fixed point by Lemma 3.3. Therefore, we may assume for the remainder of the proof that $|\lambda_1| > 1 \geq |\lambda_2|$.

Suppose φ has potential multiplicative reduction. It follows from Proposition 3.1 that φ must have three distinct fixed points, and from Proposition 3.4 we must have $\widetilde{\lambda_2} \notin \{\widetilde{0}, \widetilde{1}\}$. Using the explicit formula for λ_3 in Lemma 3.2, together with the fact that $|\lambda_2| = 1$, we find that $|\lambda_3| = 1$. Therefore $(1/\lambda_1)\sigma_1$ and $(1/\lambda_1)\sigma_2$ lie in $\mathcal{O}$; reducing modulo $\mathfrak{m}$ yields

$$\begin{aligned}\widetilde{\left(\frac{\sigma_1}{\lambda_1}\right)} &= \widetilde{1} + \widetilde{\left(\frac{\lambda_2}{\lambda_1}\right)} + \widetilde{\left(\frac{\lambda_3}{\lambda_1}\right)} = \widetilde{1}\,;\\ \widetilde{\left(\frac{\sigma_2}{\lambda_1}\right)} &= \widetilde{\lambda_2} + \widetilde{\lambda_3} + \widetilde{\left(\frac{\lambda_2\lambda_3}{\lambda_1}\right)} = \widetilde{\lambda_2} + \widetilde{\lambda_3}.\end{aligned}$$

Thus

$$\widetilde{\mathbf{s}([\varphi])} = [\widetilde{1} : \widetilde{\lambda_2} + \widetilde{\lambda_3} : \widetilde{0}].$$

Once again using the formula for λ_3 from Lemma 3.2, we have

$$\lambda_3 = \frac{1 + \lambda_2/\lambda_1 - 2/\lambda_1}{\lambda_2 - 1/\lambda_1}.$$

Reducing modulo $\mathfrak{m}$, we therefore have $\widetilde{\lambda_3} = \widetilde{\lambda_2}^{-1}$. Letting $\widetilde{x} = \widetilde{\lambda_2} + \widetilde{\lambda_2}^{-1}$ and noting that $\widetilde{\lambda_2} \neq \widetilde{1}$ implies $\widetilde{x} \neq \widetilde{2}$ completes the proof of (B).

Now suppose that φ has potential additive reduction. In the case that φ has a multiple fixed point, it follows from Proposition 3.1 and our assumption $|\lambda_1| > 1$ that $\lambda_2 = \lambda_3 = 1$. Thus $\widetilde{(\sigma_1/\lambda_1)} = \widetilde{1}$ and $\widetilde{(\sigma_2/\lambda_1)} = \widetilde{2}$. Similarly, if φ has three distinct fixed points, then by Proposition 3.4 and Lemma 3.2 we have $\widetilde{\lambda_2} = \widetilde{1}$ and $\widetilde{\lambda_3} = \widetilde{1/\lambda_2} = \widetilde{1}$. Here again we have $\widetilde{(\sigma_1/\lambda_1)} = \widetilde{1}$ and $\widetilde{(\sigma_2/\lambda_1)} = \widetilde{2}$. Thus in every case, we have

$$\widetilde{\mathbf{s}([\varphi])} = [\widetilde{1} : \widetilde{2} : \widetilde{0}]$$

as asserted.

Finally suppose that φ has potential constant reduction. It follows from Proposition 3.1 that φ must have three distinct fixed points, and from Proposition 3.4 we must have $\widetilde{\lambda_2} = 0$. Since $\widetilde{\lambda_2} = 0$, we have $|\lambda_2| < 1$, and therefore $|\lambda_3| > 1$ by Lemma 3.3. We now observe that

$$\begin{aligned}\left|\frac{\sigma_1}{\lambda_1\lambda_3}\right| &= \left|\frac{1}{\lambda_3} + \frac{\lambda_2}{\lambda_1\lambda_3} + \frac{1}{\lambda_1}\right| < 1;\\ \left|\frac{\sigma_2}{\lambda_1\lambda_3}\right| &= \left|\frac{\lambda_2}{\lambda_3} + 1 + \frac{\lambda_2}{\lambda_1}\right| = 1.\end{aligned}$$

Therefore

$$\widetilde{\mathbf{s}([\varphi])} = \left[\widetilde{\left(\frac{\sigma_1}{\lambda_1\lambda_3}\right)} : \widetilde{\left(\frac{\sigma_2}{\lambda_1\lambda_2}\right)} : \widetilde{\left(\frac{1}{\lambda_1\lambda_3}\right)}\right] = [\widetilde{0} : \widetilde{1} : \widetilde{0}],$$

as claimed. □

4.2 An application to repelling periodic points

We now use the main theorem to prove a special case of a conjecture of Hsia. For a rational map $\varphi \in K(z)$, let $\mathcal{J}_\varphi(K)$ denote the (classical) Julia set of φ, which is the complement of the equicontinuity locus (in $\mathbb{P}^1(K)$) of the family of iterates $\{\varphi^n : n \in \mathbb{N}\}$ ([11, §5.4]). Let $\mathcal{R}_\varphi(K)$ denote the set of all classical repelling periodic points for φ, and let $\overline{\mathcal{R}_\varphi(K)}$ be its closure in $\mathbb{P}^1(K)$.

It is known over the complex numbers that $\mathcal{J}_\varphi(\mathbb{C}) = \overline{\mathcal{R}_\varphi(\mathbb{C})}$; the analagous result is not known when K is non-Archimedean, though it is conjectured to be true:

Conjecture 4.1 (Hsia [7, Conjecture 4.3]) *Let φ be a rational function defined over a non-Archimedean field with* $\deg \varphi \geq 2$. *Then $\mathcal{J}_\varphi(K) = \overline{\mathcal{R}_\varphi(K)}$.*

This conjecture has not yet been resolved in general. Using part (A) of Theorem 1.1—which we recall is Theorem 3.0.3 in Yap's thesis [14]—we show that Hsia's conjecture holds for a quadratic rational map over a complete, algebraically closed non-Archimedean field:

Proposition 1.2 *Let φ be a quadratic rational map defined over K. Then $\mathcal{J}_\varphi(K) = \overline{\mathcal{R}_\varphi(K)}$.*

Proof We separate the proof based on whether or not φ has potential good reduction.

If φ has potential good reduction, the Berkovich Julia set is a single point in $\mathbf{H}^1_K$ (see [4, Proposition 0.1]); thus φ has no type I repelling periodic points (such would necessarily be Julia), hence $\mathcal{J}_\varphi(K) = \emptyset = \overline{\mathcal{R}_\varphi(K)}$ as desired.

If φ does not have potential good reduction, then by Theorem 1.1 the image $\mathbf{s}([\varphi]) \in \mathbb{A}^2(K) \subset \mathbb{P}^2(K)$ cannot specialize to $\mathbb{A}^2(k)$. It follows that φ must have a type I repelling fixed point, for if all of its type I fixed points were non-repelling then the symmetric functions in their multipliers would lie in $\mathcal{O}$. By a theorem of Bézivin ([3, Théorème 3]), this implies that $\mathcal{J}_\varphi(K) = \overline{\mathcal{R}_\varphi(K)}$. □

The argument above shows that for quadratic maps, if φ does not have potential good reduction, then φ must have a classical repelling fixed point. Such an argument cannot be used to prove Hsia's conjecture for φ of arbitrary degree, since there are examples (due to Favre and Rivera-Letelier [5]) of functions φ whose classical Julia set is empty, but whose Berkovich Julia set is a segment contained in $\mathbf{H}^1_K$ (for instance, Lattès maps associated to Tate elliptic curves; see [1, Example 10.124]). Such φ do not have good reduction, but neither they nor their iterates can have classical repelling fixed points, since such points would belong to the Julia set.

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References

1. Baker, M., Rumely, R.: AMS surveys and monographs. Potential theory and dynamics on the Berkovich projective line. American Mathematical Society, Providence (2010)
2. Berkovich, V.G.: AMS mathematical surveys and monographs. Spectral theory and analytic geometry over non-archimedean fields, vol. 33. American Mathematical Society, Providence (1990)
3. Bézivin, J.-P.: Sur les points périodiques des applications rationelles en dynamique arithmetique. Acta. Arith. **100**, 63–74 (2001)

4. Favre, C., Rivera-Letelier, J.: Théorème d'équidistribution de Brolin en dynamique *p*-adique. C. R. Math. Acad. Sci. Paris **339**(4), 271–276 (2004)
5. Favre, C., Rivera-Letelier, J.: Théorie ergodique des fractions rationelles sur un corps ultramétrique. Proc. Lond. Math. Soc. **100**, 116–154 (2010)
6. Harvey, E., Milosevic, M., Rumely, R., Watson, L.: Configuration of the crucial set for a cubic polynomial. **(in preparation)**
7. Hsia, L.-C.: Closure of periodic points over a non-Archimedean field. J. Lond. Math. Society **62**, 685–700 (2000)
8. Milnor, J.: Geometry and dynamics of quadratic rational maps. Exp. Math. **2**, 27–83 (1993)
9. Rumely, R.: The geometry of the minimal resultant locus. Preprint available at arXiv:1402.6017
10. Rumely, R.: The minimal resultant locus. Acta. Arithmetica. **169**, 251–290 (2015)
11. Silverman, J.: The arithmetic of dynamical systems, GTM 241. Springer, New York (2007)
12. Silverman, J.: Moduli spaces and arithmetic dynamics. American Mathematical Society, Providence (2012)
13. Silverman, J.: The space of rational maps on $\mathbb{P}^1$. Duke Math. J. **94**, 41–77 (1998)
14. Yap, D.: Potential good reduction of degree two rational maps. Doctoral Thesis, University of Hawaii at Manoa (2012)